Squeeze! Don't Move!

Tight Configuration of Disks and their Circum-rectangle

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Abstract

In this project, we study the "***tight configurations***" for n disks of the same size in their *circum-rectangles* and find the biggest and smallest such rectangles when $n \leq 6$. We also find the smallest rectangle for arbitrary n of certain configurations and discover several methods for generating interesting tight configurations of ***any*** number of disks based on simple ones.

Introduction

In figure 1, several disks of the same size form a tight configuration in their "***circum-rectangle***". By "***tight***", we mean the disks are not able to move.

Our motivation is "*How to make the biggest or smallest circum-rectangle for a given number of disks?*" We partially answer this question and find several ways to construct tight configurations of any given number of disks.

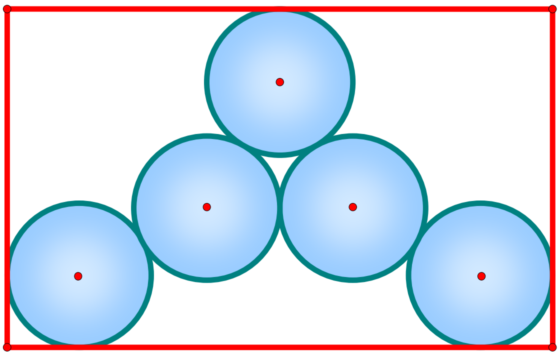


Figure 1. Tight configuration

Definitions

1. **Tight configuration:** A configuration is said to be tight if the disks are not able to move when given any direction of force.
2. **Circum-rectangle:** The rectangle which the disks remain tight in.
3. **Free configuration**: A configuration is said to be free if the disks move when given forces.

Equipments

Geometer's Sketchpad, WxMaxima, Graphmatica, Excel, paper, pens, coins.

Goals

1. Find the circum-rectangles with extreme areas of a few disks.

2. Compare the areas of circum-rectangles in linear and equilateral triangle configuration.

3. Construct tight configurations with a large number of disks.

Procedure

Section 1- Find the circum-rectangles with extreme areas of a few disks.

Limits of tight configuration:

The arcs formed by adjacent tangent points have to be smaller than $180°$. Also, the disks can not rotate together to keep the disks tight.

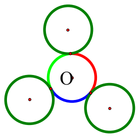


Figure2. DiskO is tight.

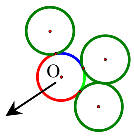


Figure 3. DiskO is free.

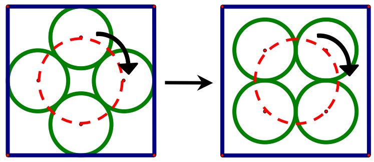
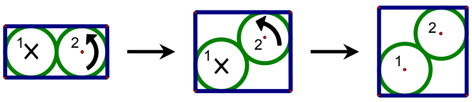


Figure 4. Free configuration.

Two disks

First, we find out all the possible tight configurations of 2 disks and get the approximation extreme areas of their circum-rectangles by using dynamic geometry mathematical software *Geometer's Sketchpad*. Then we get the extreme value of their *area functions* by Calculus.



The maximal area = $\left(2+\sqrt{2}\right)^2 \approx 16.65$

More disks

These are similar to two disks.

We compare them and find the circum-rectangles with the extreme area as shown:

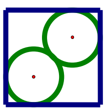
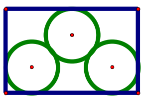
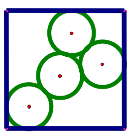
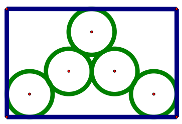
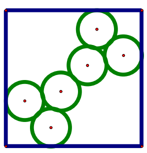
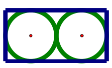
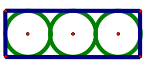
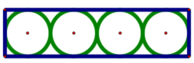
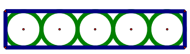
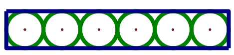
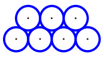
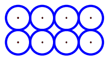
Number of disks	2	3	4	5	6
The maximal areas of circum-rectangle	$\left(2+\sqrt{2}\right)^2$ ≈ 11.66	16.65	$\left(2+\frac{3\sqrt{2}}{2}+\frac{\sqrt{6}}{2}\right)^2$ ≈ 28.58	35.51	$\left(2+2\sqrt{2}+\sqrt{6}\right)^2$ ≈ 52.97
Corresponding circum-rectangle					
The minimal areas of circum-rectangle	8	12	16	20	24
Corresponding circum-rectangle					

Table 1. The circum-rectangle with extreme areas of 2~6 disks which have radius 1.

Section 2- Compare the areas of circum-rectangles in linear and equilateral triangular configuration.

Linear and equilateral triangular configuration

Our result in *Section 1* contradicts the popular belief that an ***equilateral triangular configuration*** can always results in a smaller area than a ***linear configuration*** does. A configuration is said to be equilateral triangular if the configuration can be divided into parallel layers and two adjacent layers are connected in the ways of (*A*)  or (*B*) (with at least an (A)).

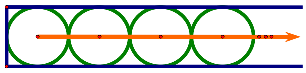
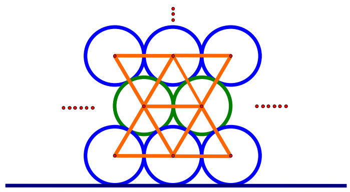
Linear configuration	***Equilateral triangular configuration***
	

Table 2.The linear configuration and the equilateral triangular configuration.

Two and three layers in the equilateral triangular configuration

Case 1 *(2k disks)*

Suppose there are n disks in each layer ($2n$ disks)

Let the equilateral triangular configuration be smaller: $(2+\sqrt{3})(2n+1) > 2\times2n$

When there are $2n$ ($n \geq 7$) disks, the equilateral triangular configuration results in a smaller area than the linear configuration does.

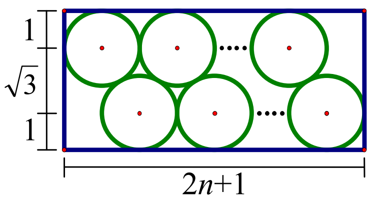


Figure 5. The equilateral triangular configuration.

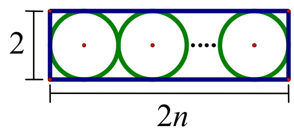


Figure 6. The linear configuration.

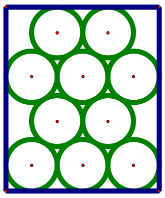
Other Cases *(two and three layers)*

These are similar to *Case 1*.

Therefore, the equilateral triangular configuration results in a smaller area than the linear configuration does when there are more than 14 disks.

Four layers and more in the equilateral triangular configuration

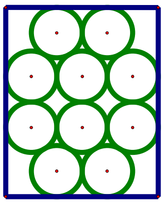
To avoid duplications, we compare the two configurations by their height and the largest number of disks. Here is an example of four layers: there are two possible configurations as shown.



Height = $6+\sqrt{3}$

Area = $6(2+3\sqrt{3})\approx 43.18$

Area of liner configuration = 40



Height = $4+2\sqrt{3}$

Area = $6(4+2\sqrt{3})\approx 44.78$

Area of liner configuration = 40

Results

1. The linear configuration results in a smaller area than the equilateral triangular configuration does when there are 2~10, 12, or 13 disks.

2. The equilateral triangular configuration results in a smaller area than that of the linear configuration when there are 11, 14 disks or more.

Section 3 - *Construct tight configurations with a large number of disks.*

Basic configurations:

A configuration is said to be ***basic*** that if it satisfies the following features:

1. The two curves formed by the most outside disks are absolute value function. One is increasing function, $f'(x)>1$. The other one is decreasing function, $1>f'(x)>0$.

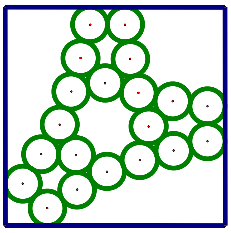


Figure 7. A basic configuration.

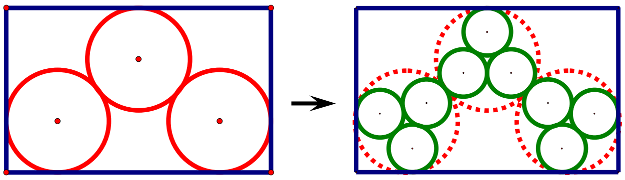


Figure 3. Placing basic configurations.

Basic configurations can be placed in a simpler configuration and construct a tight configuration with more disks. However, only ***wave*** and ***ring*** configurations are possible: W*ave* configurations can be unlimited extending. R*ing* configurations have at most eight basic configurations.

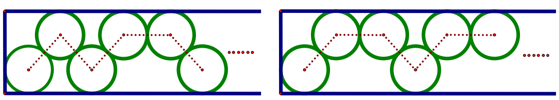


Figure 5. *Wave configurations*

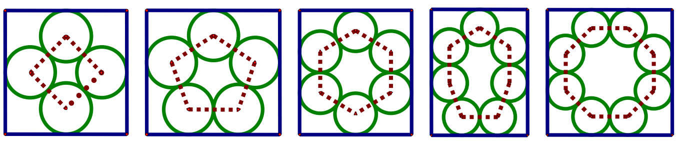
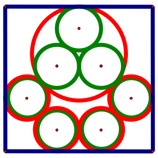
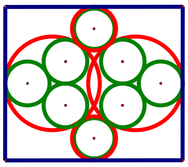
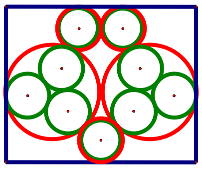
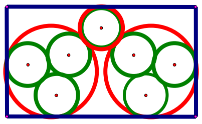
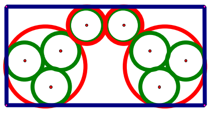
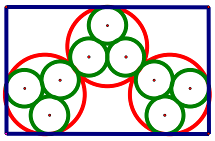
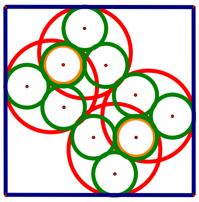
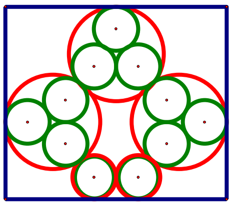
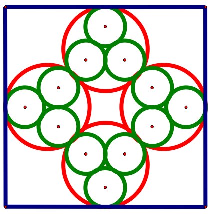
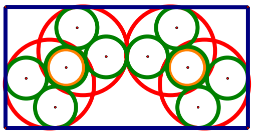
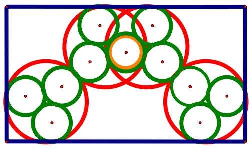
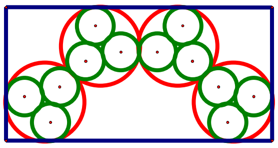
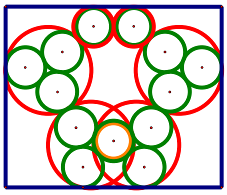
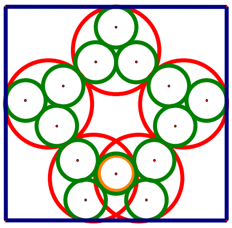
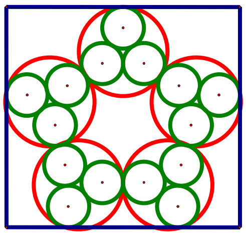
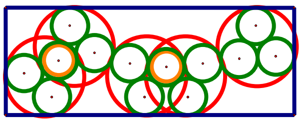
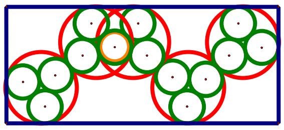
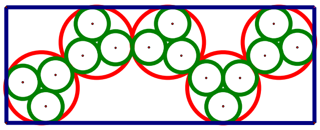
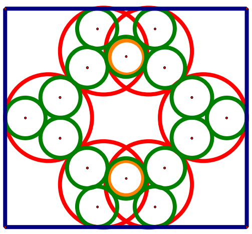
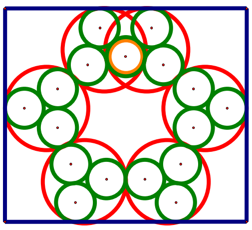
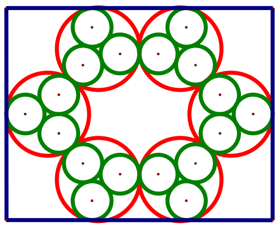
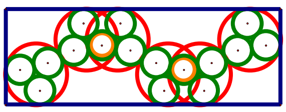
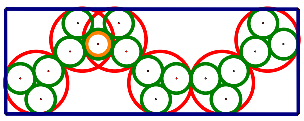
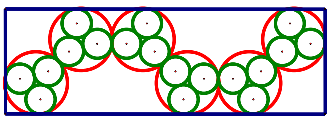
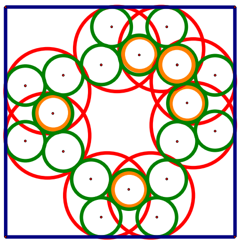
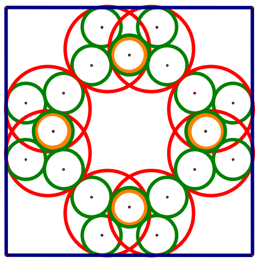
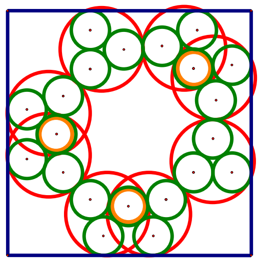
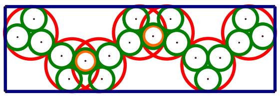
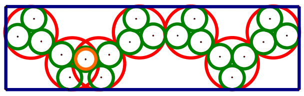
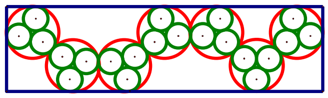
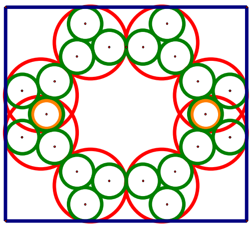
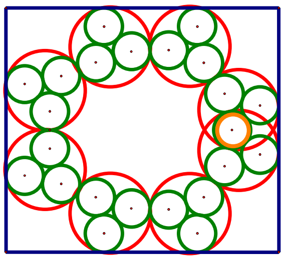
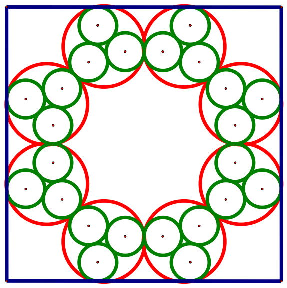
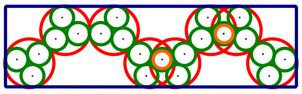
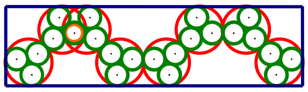
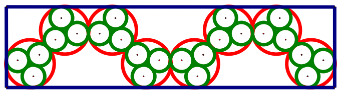


Figure 6. *Ring configurations*

Here are the results of placing the basic configurations formed by three disks in *wave* and *ring* configurations:

Number of disks	7	8	9
Constructed from a ring configuration			
Constructed from a wave configuration			
Number of disks	10	11	12
Constructed from a ring configuration			
Constructed from a wave configuration			
Number of disks	13	14	15
Constructed from a ring configuration			
Constructed from a wave configuration			
Number of disks	16	17	18
Constructed from a ring configuration			
Constructed from a wave configuration			

Number of disks	19	20	21
Constructed from a ring configuration			
Constructed from a wave configuration			
Number of disks	22	23	24
Constructed from a ring configuration			
Constructed from a wave configuration			

Basic configuration decomposition method:

Series of basic numbers

When denoting a basic configuration by the number of the disks it contains, called a *"basic number"*, a configuration constructed from a wave configuration can be represented by a series of basic numbers and the number of the overlapping-disks (negative). Here is an example:

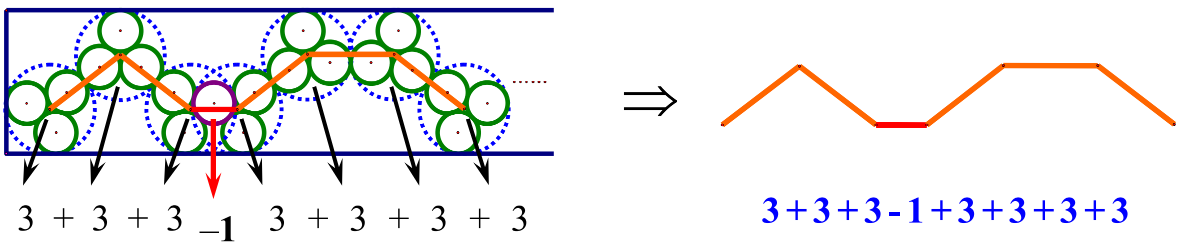
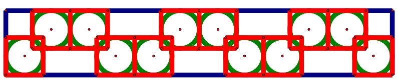
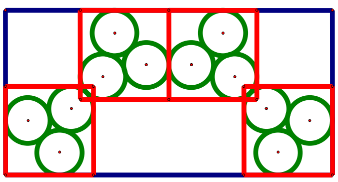
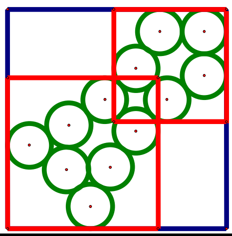
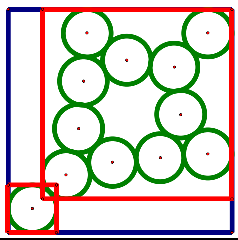


Figure 8. The series and the corresponding tight configuration.

Different tight configurations of 12 disks:

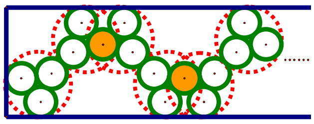
1+1+1+1+1+1+1+1+1+1+1+1	3+3+3+3	5+7	1+11
			

Construct tight configurations of any number of disks:

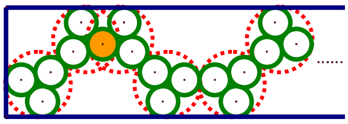
Given n = 3(2k)-s or 3(2k+1)-s ($k \geq 3, k \in N$, *s = 0, 1, 2).*

1. If *n = 3(2k)-s:*

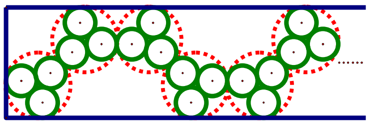
s = 0



s = 1

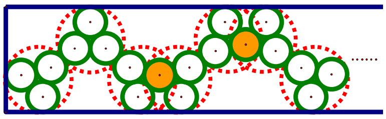


s = 2

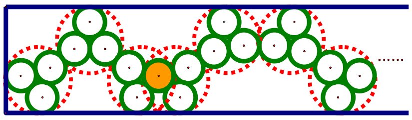


2. If *n = 3(2k+1)-s:*

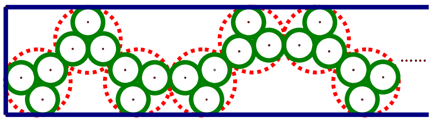
s = 0



s = 1



s = 2



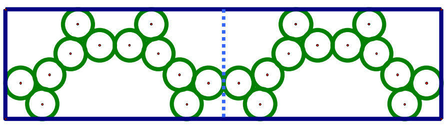
In this way, we can construct *tight* configurations of *any* number of disks.

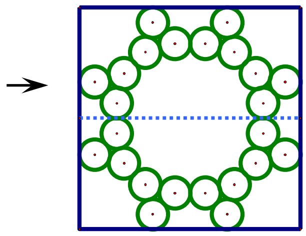
Also, we can construct more tight configurations by using different basic configurations.

Rearrangement:

However, the length of the adjacent sides of the circum-rectangle of a configuration constructed from a wave configuration can be quite different. We can reduce the length/width ratio by rearranging appropriate subconfigurations. There are two ways to rearrange basic configurations:

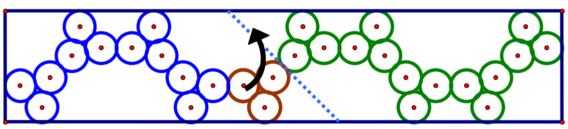
1. If the number of basic configurations is 2*n* (*n* is an even number).

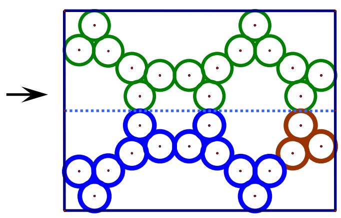




Rearrangement
(Without changing the area)

2. If the number of basic configurations is 2*n* (*n* is a composite number and has an odd divisor).

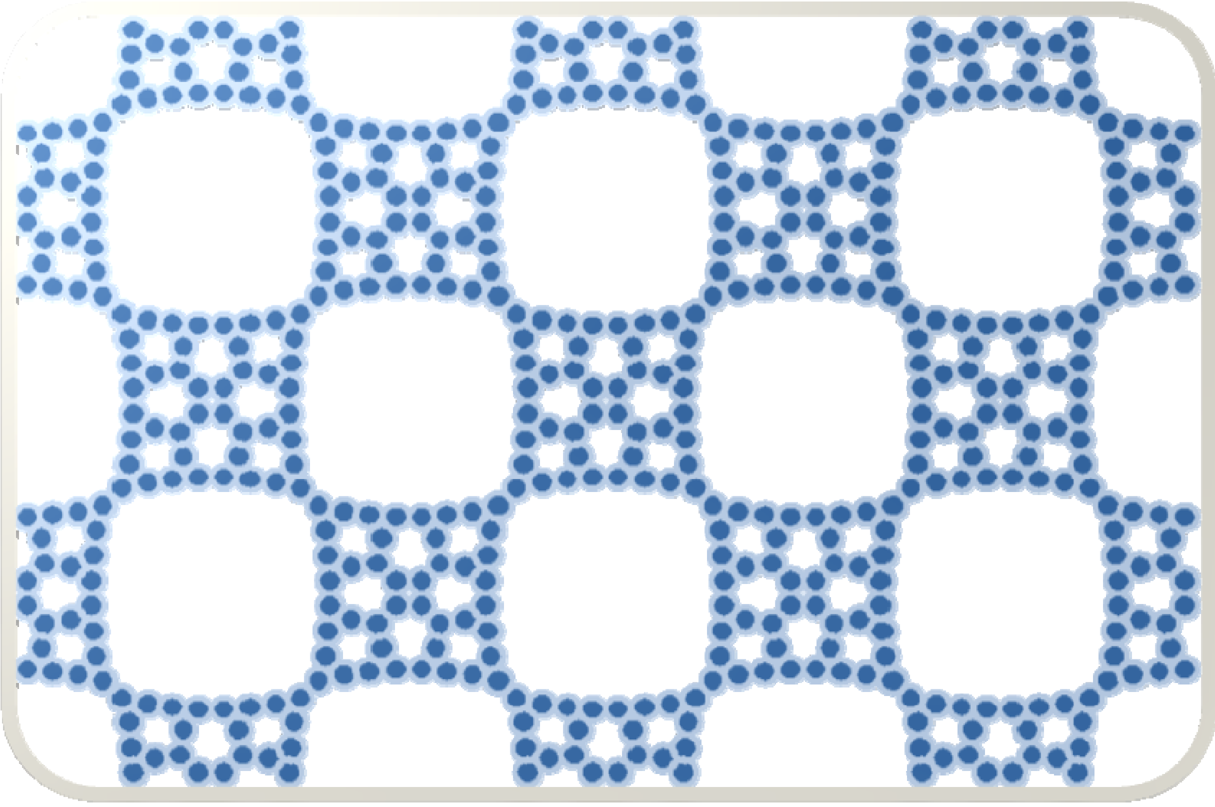


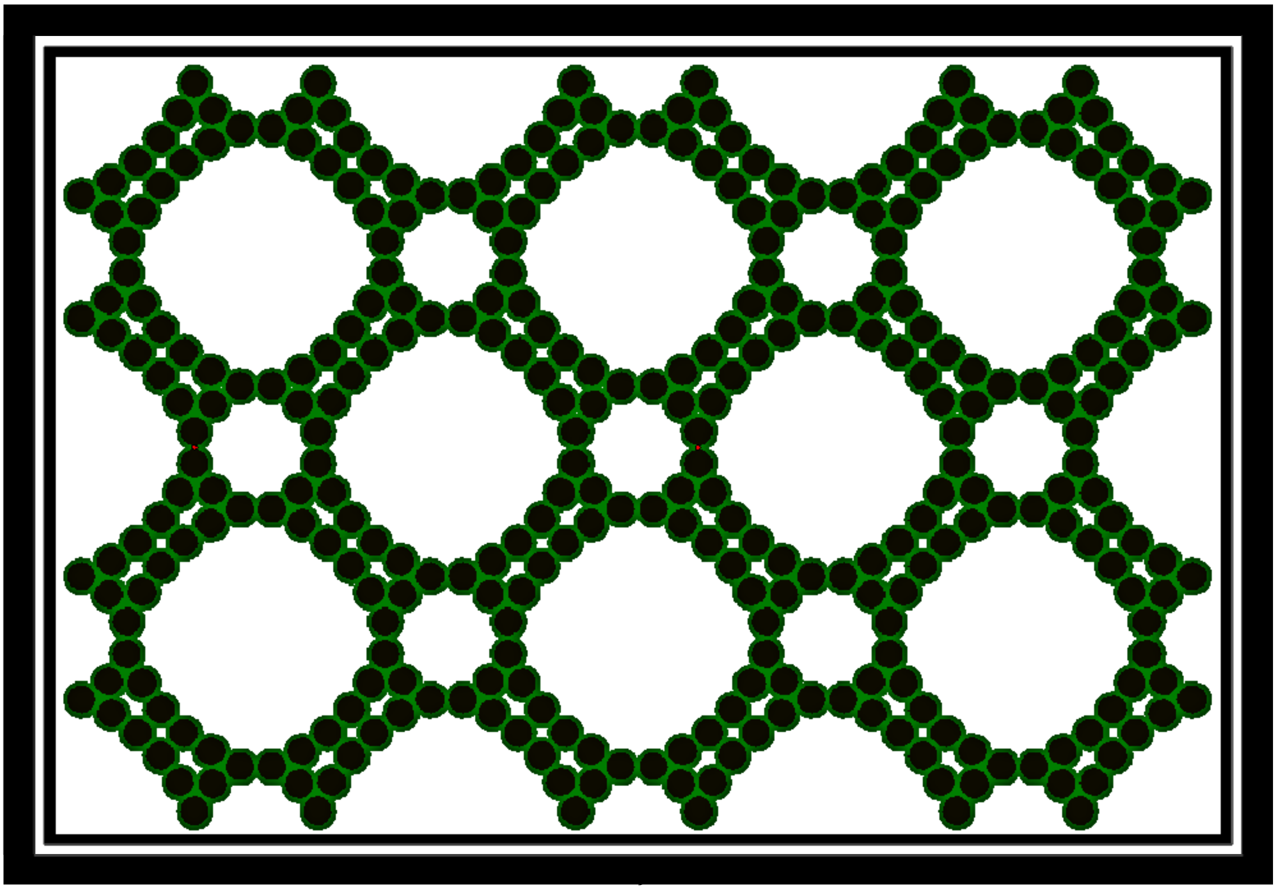


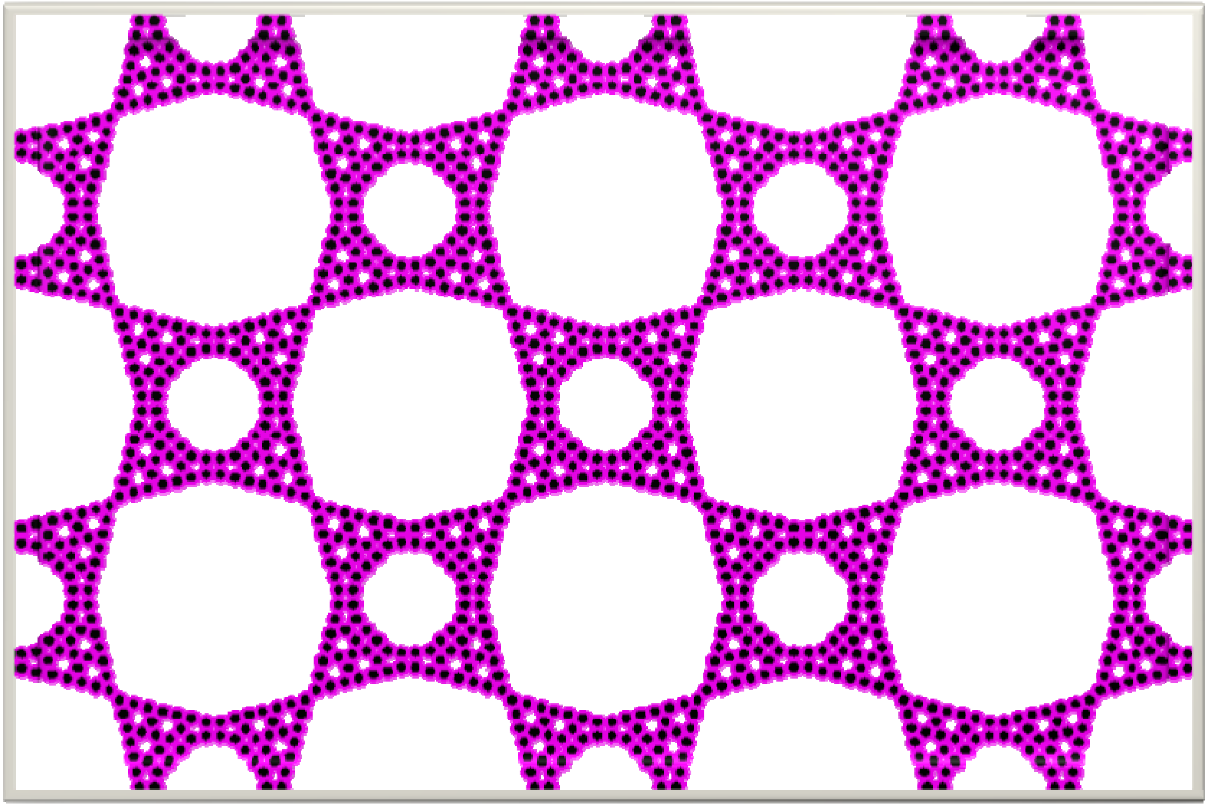
Rearrangement
(Reducing the area)

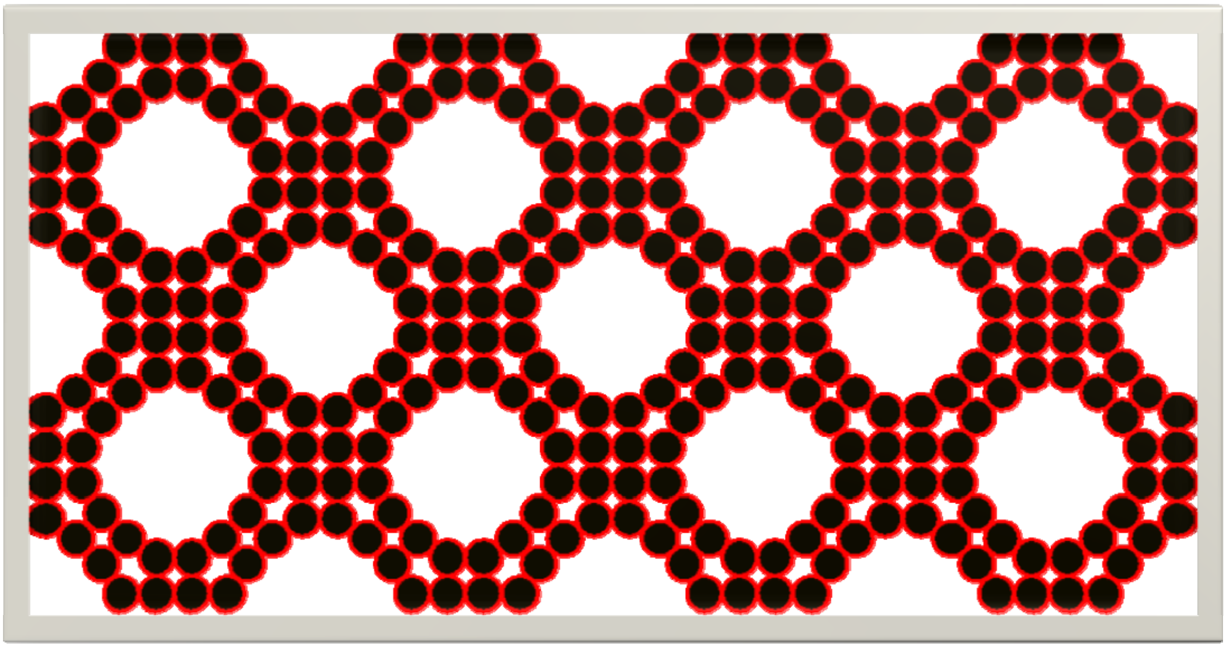
Extension:

To increase the area of a circum-rectangle, we can also extend a circum-rectangle by reflections with respect to the edges and get larger tight configurations with beautiful patterns.



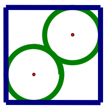
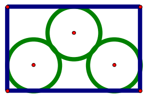
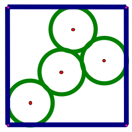
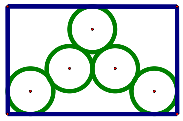
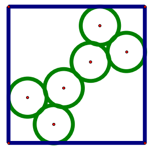
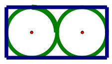
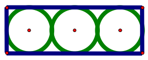
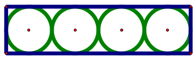
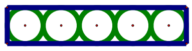
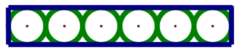






Conclusions

1. We find the circum-rectangles that achieve the maximal and minimal areas when there are 2~6 disks as follow:

Number of disks	2	3	4	5	6
The maximal areas of circum-rectangle (cm^2)	$\left(2+\sqrt{2}\right)^2$	16.65	$\left(2+\frac{3\sqrt{2}}{2}+\frac{\sqrt{6}}{2}\right)^2$	35.51	$\left(2+2\sqrt{2}+\sqrt{6}\right)^2$
Corresponding circum-rectangle					
The minimal areas of circum-rectangle (cm^2)	8.00	12.00	16.00	20.00	24.00
Corresponding circum-rectangle					

2. The linear configuration results in a smaller area than that of the equilateral triangle configuration with 2~10, 12, and 13 disks.

3. By using the *basic configuration decomposition method*, we can construct tight configurations of ***any*** number of disks.

4. For the application, our work may be applied to pattern design, or optimal arrangements of the pipes in a filter or an optical fiber.

References

1. George G. Szpiro, Kepler's Conjecture, John Wiley & Sons. Inc, 2003.

2. Ivan Moscovich, Leonardo's Mirror & Other Puzzles, Sterling Publishing, 2005.

3. Wolfram math world: http://mathworld.wolfram.com/topics/PackingProblems.html

4. Erich's Packing Center: http://www.stetson.edu/~efriedma/packing.html